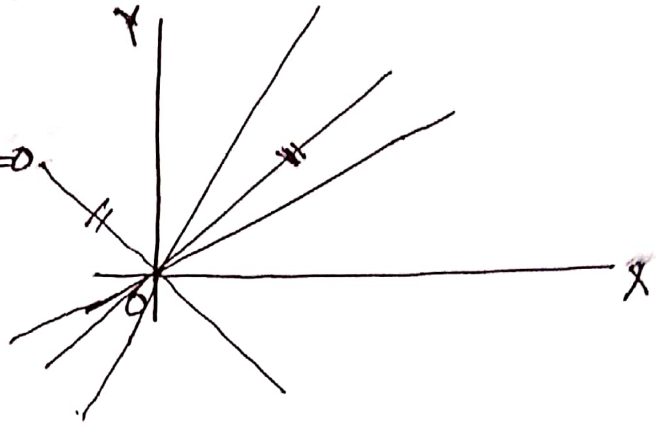
Art. 39: Equation of bisectors of the angles between the lines represented by $ax^2+2hxy+by^2=0$.



We have the lines ~~ax²+~~

$ax^2+2hxy+by^2 \equiv (y-m_1x)(y-m_2x)=0$

$\therefore y-m_1x=0$ & $y-m_2x=0$

The equ. of bisectors of the angles between them;

$$\frac{y-m_1x}{\sqrt{1+m_1^2}} = \pm\frac{y-m_2x}{\sqrt{1+m_2^2}}$$

or, $(y-m_1x)^2(1+m_2^2)=(y-m_2x)^2(1+m_1^2)$

or, $(y^2-2m_1xy+m_1^2x^2)(1+m_2^2) = (y^2-2xym_2+m_2^2x^2)(1+m_1^2)$

or, $\cancel{y^2}-2m_1xy+m_1^2x^2+m_2^2y^2-2m_1m_2^2xy+\cancel{m_1^2m_2^2x^2}$
$= \cancel{y^2}-2m_2xy+m_2^2x^2+m_1^2y^2-2m_1^2m_2xy+\cancel{m_1^2m_2^2x^2}$

or, $(m_1^2-m_2^2)x^2+(m_2^2-m_1^2)y^2 = 2(m_1-m_2)xy$
$+2m_1m_2(m_2-m_1)xy$

or, $(m_1^2-m_2^2)(x^2-y^2) = 2xy\{(m_1-m_2)-(m_1-m_2)m_1m_2\}$

or, $(m_1+m_2)(x^2-y^2)=2xy\{1-m_1m_2\}$

or, $-\frac{\cancel{2}h}{b}\cdot(x^2-y^2) = \cancel{2}xy\left(1-\frac{a}{b}\right)$

or, $\frac{h(x^2-y^2)}{-b} = \frac{(b-a)xy}{b}$

or, $\frac{h(x^2-y^2)}{-\cancel{b}} = \frac{(a-b)xy}{-\cancel{b}}$

or, $\frac{x^2-y^2}{a-b}=\frac{xy}{h}$; which are the equations of bisectors.

Q. Find the condition that the general equation of 2nd degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of st. lines.

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{———} (1)$$

If we transfer the origin to the point (α,β), then (1) becomes $a(x+\alpha)^2+2h(x+\alpha)(y+\beta)+b(y+\beta)^2+2g(x+\alpha)$
$$+2f(y+\beta)+c=0$$

$$ax^2+2a\alpha x+a\alpha^2+2h(xy+\alpha y+\beta x+\alpha\beta)+by^2$$
$$+2b\beta y+b\beta^2+2gx+2g\alpha+2fy+2f\beta+c=0$$

$$ax^2+2hxy+by^2+2(a\alpha+h\beta+g)x+2(h\alpha+b\beta+f)y$$
$$+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \text{———} (2)$$

Equation (2) will represent a pair of st. lines if

$$a\alpha+h\beta+g=0 \quad \text{———} (3)$$
$$h\alpha+b\beta+f=0 \quad \text{———} (4)$$
$$\& \quad a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \text{———} (5)$$

$$(5) \Rightarrow \quad \alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+g\alpha+f\beta+c=0$$

or, $\alpha\cdot 0+\beta\cdot 0+g\alpha+f\beta+c=0$ [by (3) & (4)]

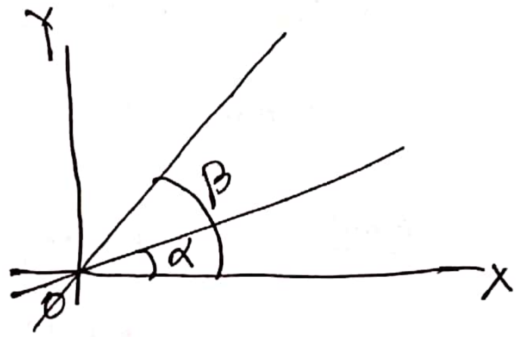
or, $g\alpha+f\beta+c=0 \quad \text{———} (6)$

If we eliminate α,β from the eqns. (3), (4) & (6)

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

or, $abc+2fgh-af^2-bg^2-ch^2=0$ which is the required condition.

Q. 17 If the lines $x^2(\tan^2\varphi+\cos^2\varphi)-2xy\tan\varphi+y^2\sin^2\varphi=0$ makes angles α and β with the axis of x, show that $\tan\alpha-\tan\beta=2$.



Ans:- We have the st. lines

$$x^2(\tan^2\varphi+\cos^2\varphi)-2xy\tan\varphi+y^2\sin^2\varphi=0$$

$$\left(\frac{\sin^2\varphi}{\cos^2\varphi}+\cos^2\varphi\right)x^2-2xy\cdot\frac{\sin\varphi}{\cos\varphi}+y^2\sin^2\varphi=0$$

$$(\sin^2\varphi+\cos^4\varphi)x^2-2\sin\varphi\cos\varphi\, xy+y^2\sin^2\varphi\cos^2\varphi\equiv(y-m_1x)\cdot(y-m_2x)=0$$

Equating the coefficients;

Here $\tan\alpha=m_1$, $\tan\beta=m_2$

$\sin^2\varphi+\cos^4\varphi=m_1m_2$;

$\cancel{-}2\sin\varphi\cos\varphi=\cancel{-}(m_1+m_2)$

& $\sin^2\varphi\cos^2\varphi=1$.

Now $\cancel{(m_1+m_2)^2}=\cancel{4\sin^2\varphi\cos^2\varphi}$

$\cancel{(m_1-m_2)^2+4m_1m_2}=\cancel{4\sin^2\varphi\cos^2\varphi}$

Now $(m_1-m_2)^2=(m_1+m_2)^2-4m_1m_2$

$=4\sin^2\varphi\cos^2\varphi-4(\sin^2\varphi+\cos^4\varphi)$

$=4(\sin^2\varphi\cos^2\varphi-\sin^2\varphi-\cos^4\varphi)$

$=4(1-\sin^2\varphi-\cos^4\varphi)$

$=4(\cos^2\varphi-\cos^4\varphi)$

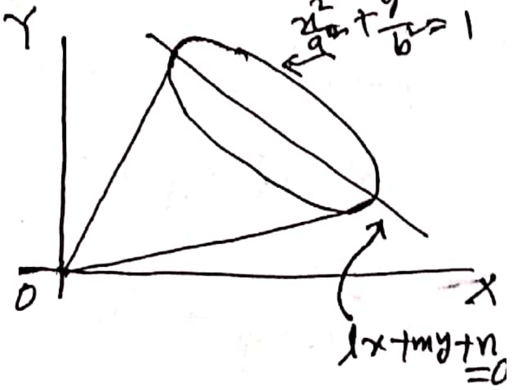
$=4\cos^2\varphi(1-\cos^2\varphi)=4\cos^2\varphi\sin^2\varphi$

$=4$

$\therefore\ m_1-m_2=2$

$\tan\alpha-\tan\beta=2$ Proved.

Q.19 Prove that the pair of st. lines joining the origin to the points of intersection of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ by the line $lx+my+n=0$ are coincident if $a^2l^2+b^2m^2=n^2$.



Ans: $lx+my+n=0$ given line.

$$lx+my=-n$$

$$\frac{lx+my}{-n}=1$$

Now the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{lx+my}{-n}\right)^2$$

or, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{l^2x^2+2lmxy+m^2y^2}{n^2}$

or, $\left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)x^2-\frac{2lm}{n^2}xy+\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right)y^2=0$ —— (i)

For coincidence; $h^2-ab=0$

$$\boxed{\text{In (i); } h=-\frac{lm}{n^2};\ a=\frac{1}{a^2}-\frac{l^2}{n^2},\ b=\frac{1}{b^2}-\frac{m^2}{n^2}}$$

$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}$

when $\theta=0$

$\tan 0=\frac{2\sqrt{h^2-ab}}{a+b}$

$0=\frac{2\sqrt{h^2-ab}}{a+b}$

$2\sqrt{h^2-ab}=0$

or, $h^2-ab=0$

$$\left(-\frac{lm}{n^2}\right)^2=\left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right)$$

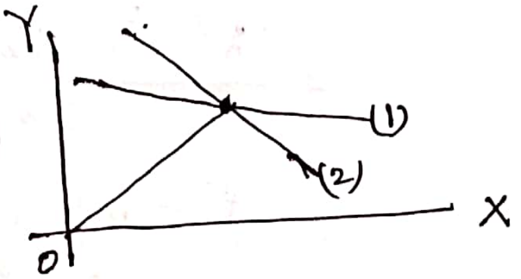
$$\frac{l^2m^2}{\cancel{n^4}}=\frac{n^2-a^2l^2}{a^2\cancel{n^2}}\cdot\frac{n^2-b^2m^2}{b^2\cancel{n^2}}$$

$$\cancel{a^2b^2l^2m^2}=n^4-a^2l^2n^2-b^2m^2n^2+\cancel{a^2b^2l^2m^2}$$

or, $a^2l^2n^2+b^2m^2n^2=n^4$

or, $a^2l^2+b^2m^2=n^2$. Proved.

Q.29 If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two st. lines, prove that the square of the distance of their point of intersection from the origin is $\dfrac{c(a+b)-f^2-g^2}{ab-h^2}$.



Ans: Let the lines represented by the equations $lx+my+n=0$ (1) & $l'x+m'y+n'=0$ (2) then

$$ax^2+2hxy+by^2+2gx+2fy+c \equiv (lx+my+n)(l'x+m'y+n')=0$$

Equating the coefficients & constant terms;

$$ll'=a;\ lm'+l'm=2h;\ mm'=b;\ ln'+l'n=2g$$
$$nm'+n'm=2f$$
$$nn'=c.$$

Now, to find the point of intersection, we have solve (1) & (2);

$$lx+my+n=0$$
$$l'x+m'y+n'=0$$

$$\therefore \quad \frac{x}{mn'-m'n}=\frac{y}{l'n-ln'}=\frac{1}{lm'-l'm}$$

$$\therefore \quad x=\frac{mn'-m'n}{lm'-l'm}\ ;\quad y=\frac{l'n-ln'}{lm'-l'm}$$

$\therefore$ point of intersection $\left(\dfrac{mn'-m'n}{lm'-l'm}, \dfrac{l'n-ln'}{lm'-l'm}\right)$.

Square of the distance from the origin $(0,0)$ to the point of intersection is $=\left(0-\dfrac{mn'-m'n}{lm'-l'm}\right)^2+\left(0-\dfrac{l'n-ln'}{lm'-l'm}\right)^2$

$$=\left(\frac{mn'-m'n}{lm'-l'm}\right)^2+\left(\frac{l'n-ln'}{lm'-l'm}\right)^2$$

$$= \frac{\{(mn' + m'n)^2 - 4mm'nn'\} + \{(ln' + l'n)^2 - 4ll'nn'\}}{(lm' + l'm)^2 - 4ll'mm'}$$

$$= \frac{(2f)^2 - 4bc + (2g)^2 - 4ac}{(2h)^2 - 4ab}$$

$$= \frac{f^2 - bc + g^2 - ac}{h^2 - ab}$$

$$= \frac{bc + ac - f^2 - g^2}{ab - h^2}$$

$$= \frac{c(a+b) - f^2 - g^2}{ab - h^2} \quad \text{. proved .}$$

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